Pergamon

Int. J. Heat Mass Transfer. Vol. 39, No. 4, pp. 823–830, 1996

0017–9310/96 $15.00+0.00

0017–9310(95)00167–0

A boundary integral equation method for nonlinear heat conduction problems with temperature-dependent material properties

TAKANOBU GOTO and MASAAKI SUZUKI

Research Laboratory for Nuclear Reactors, Tokyo Institute of Technology, 2-12-1, O-okayama, Meguro-ku, Tokyo 152, Japan

(*Received* 21 *December* 1994 *and in final form* 27 *March* 1995)

Abstract—A boundary integral equation formulation for transient heat conduction problems with temperature-dependent material properties is proposed. Direct regular method, a technique used in elastostatic boundary element analysis, is applied for numerical solution. In order to verify the validity of the present formulation, two problems are solved. Present results show good agreement with the exact solution and the results obtained by finite difference methods and finite element method. To evaluate the coefficients of discretized boundary integral equations, some integral formulae are used instead of numerical quadrature. Application of these formulae can reduce the computation time to a large extent and achieve highly accurate evaluation.

1. INTRODUCTION

Analyses of nonlinear heat conduction problems become more important and of more interest in various engineering and science fields, since the studies of the phenomena under more severe and more complicated conditions are required. Various kinds of nonlinearities appear in heat conduction problems, for example, nonlinear heat source and sink, nonlinear boundary conditions such as radiation and nonlinear convective heat transfer on boundary, heat conduction with phase change and temperature-dependent material properties. To solve these nonlinear problems, finite difference methods (FDM), finite element methods (FEM) and control volume method have been used commonly [1–4]. Boundary element method (BEM), which was found to be valid and efficient for linear heat conduction problems [5, 6], has also been investigated in connection with the applications to nonlinear problems. Among those nonlinear problems, nonlinear heat sink and source, nonlinear boundary conditions and phase change problems have been studied and solved numerically [7–10], whereas nonlinear heat conduction problems with temperature-dependent material properties, which are important in the case that the variation of material properties with temperature can not be neglected in the range of temperature concerned, were studied not from the numerical point of view, but just by linear modellings [7, 11, 12]. The results obtained from the linearized models are approximate solutions of the problems, but could not reflect the nonlinear characteristics that the original problems have. The solutions of linearized models may not be able to provide us with the information required. Compared with FDM, FEM and control volume method, which have the techniques developed to solve this type of nonlinear problems, the lack of the techniques for these problems in BEM seems to be a defect of it.

In the present study, a formulation of BEM for this kind of nonlinear heat conduction problems is proposed in order to resolve the disadvantage mentioned above. It is easy to be implemented numerically and does not require any modellings. Derived boundary integral equation is discretized by usual boundary element techniques and the direct regular method (DRM). DRM is a technique that source points are located not on the boundary but outside the domain [13, 14]. By applying DRM, significant improvement of accuracy of solution is achieved. In order to confirm the validity of the present boundary integral formulation, two examples are solved. The first one is a problem whose exact solution is known, and the second is the one solved by some other methods [15]. Numerical results obtained by the present method show good agreement with the exact solution and the results reported by other authors. The comparison of the results proves the potential of the present boundary integral equation method (BIEM).

In BEM and BIEM analysis of heat conduction problems, numerical quadratures are employed for the integrations of the fundamental solution to evaluate the coefficients of discretized equation system. This process consumes a large amount of computation time. In the present analysis, we use, instead of numerical quadrature, some integral formulae of the integrations of the fundamental solution of two-dimensional (2D) heat equation [16]. Application of these formulae can reduce computation time for the above process to a great extent and achieve the highly

NOMENCLATURE

$c(T)$	specific heat
C_r	a reference value of transformed specific heat
$H(u)$	defined in equation (9)
k	thermal conductivity
n	unit outer normal
$\mathbf{r}$	position vector
T	temperature
t	time
u	Kirchhoff's-transformed temperature
x	horizontal coordinate
y	vertical coordinate.

Greek symbols

α	thermal diffusivity
β	temperature coefficient for the specific heat
Γ	boundary of Ω
γ	Euler's constant
Δ	Laplacian operator
ε	absolute error
κ	temperature coefficient for the thermal conductivity
ρ	density
τ	time step
Ω	domain under consideration.

Subscripts

b	bottom
l	left
i	source point
r	right
t	top.

Superscripts

exact	exact solution
num	numerical solution
*	fundamental solution.

accurate evaluations. As a result of it, the efficiency and the accuracy of the solution can be improved, and the re-calculation of the coefficients is shown to be practicable, which has seemed to be inefficient and unrealistic with numerical quadrature so far.

2. FORMULATION

The heat conduction equation with temperature-dependent thermal conductivity, density and specific heat in the form,

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \nabla\cdot(k(T)\nabla T) \tag{1}$$

is considered. For the modification of this equation, Kirchhoff's transformation [11]

$$u(T) = \int_{T_0}^{T} k(s)\,\mathrm{d}s \tag{2}$$

where T_0 is an arbitrary constant, is employed. By performing this transformation, equation (1) is transformed to

$$\frac{\rho(T)c(T)}{k(T)}\frac{\partial u}{\partial t} = \Delta u. \tag{3}$$

Defining $C(u) = \rho(T)c(T)/k(T)$ and $\alpha(u) = 1/C(u)$, the above equation is rewritten as

$$C(u)\frac{\partial u}{\partial t} = \Delta u \tag{4}$$

or equivalently

$$\frac{\partial u}{\partial t} = \alpha(u)\Delta u \tag{5}$$

where, of course, $C(u)$ and $\alpha(u)$ may be assumed to be positive for all actual heat conduction problems. In order to derive a boundary integral equation from the equation (4), we employ the fundamental solution of the equation,

$$C_r\frac{\partial u}{\partial t} + \Delta u = 0 \tag{6}$$

where C_r is a reference value of $C(u)$. We will discuss the choice of this constant later in Section 4. In the 2D case, the fundamental solution is

$$u^*(r,t;r_i,t_i) = \frac{C_r}{4\pi(t_i-t)}\exp\left(-\frac{C_r(r-r_i)^2}{4(t_i-t)}\right). \tag{7}$$

Using u^*, a boundary integral equation is obtained as

$$c_i u_i = \int_0^\tau\int_\Gamma\left(\frac{\partial u}{\partial n}u^* - \frac{\partial u^*}{\partial n}u\right)\mathrm{d}\Gamma\,\mathrm{d}t + \int_\Omega [uC_r u^*]_{t=0}\,\mathrm{d}\Omega$$
$$-\int_0^\tau\int_\Omega (C(u)-C_r)\frac{\partial u}{\partial t}u^*\,\mathrm{d}\Omega\,\mathrm{d}t \tag{8}$$

where c_i is a function of the location of source point and the geometry of the boundary [5].

In order to transform the nonlinear domain integral term on the right-hand side of equation (8) into the form convenient to treat numerically, we introduce a function [16, 17],

$$H(u) = \int_{u_0}^{u}(C(s)-C_r)\,\mathrm{d}s \tag{9}$$

where u_0 is an arbitrary constant. We have its derivative with respect to t as

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial t} = (C(u) - C_r)\frac{\partial u}{\partial t}. \quad (10)$$

Substituting equation (10) into the nonlinear integral term, we have

$$\int_0^\tau \int_\Omega (C(u) - C_r)\frac{\partial u}{\partial t} u^* \, d\Omega \, dt = \int_0^\tau \int_\Omega \frac{\partial H(u)}{\partial t} u^* \, d\Omega \, dt. \quad (11)$$

Let the time discretization of BEM be recalled here for further reduction. We employ the constant element in time in the present formulation. This discretization postulates that the unknowns are constant in the semi-open time interval (0, τ], namely, they are assumed to vary with time as shown in Fig. 1, and yield the fully implicit scheme [4, 5]. This postulate leads the following approximation of the right-hand side of equation (11) as

$$\begin{aligned}\int_0^\tau \frac{\partial H(u)}{\partial t} u^* \, dt &= [H(u)u^*]_{t=0}^{t=\tau} - \int_0^\tau H(u)\frac{\partial u^*}{\partial t} dt \\ &\approx [H(u)u^*]_{t=0}^{t=\tau} - [H(u)]_{t=\tau}[u^*]_{t=0}^{t=\tau} \\ &= [H(u)]_{t=0}^{t=\tau}[u^*]_{t=0} \\ &= [H(u(r,\tau)) - H(u(r,0))]u^*(r,0;r_i,\tau). \end{aligned} \quad (12)$$

Using this approximation, the boundary integral equation (8) is rewritten as

$$\begin{aligned} c_i u_i = &\int_0^\tau \int_\Gamma \left(\frac{\partial u}{\partial n}u^* - \frac{\partial u^*}{\partial n}u\right) d\Gamma \, dt \\ &+ \int_\Omega [C_r u(r,0) - H(u(r,\tau)) \\ &+ H(u(r,0))]u^*(r,0;r_i,\tau) \, d\Omega. \end{aligned} \quad (13)$$

It should be noted that the nonlinear domain integral term of this equation has been time-integrated and does not contain $\partial u/\partial t$. The boundary integral term is to be integrated with respect to time by usual BEM technique.

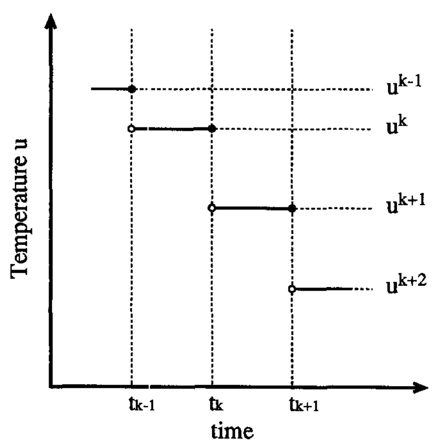

Fig. 1. Variation of temperature with time for boundary element scheme.

3. DISCRETIZATION

In the present paper, 2D problems are solved by the present method. It is, however, easily seen that no 2D particular property is used and the 3D application is straightforward.

The boundary integral equation (13) is discretized with linear continuous boundary and internal elements in space, and with constant elements in time as mentioned in the previous section. Double nodes are employed at the corners of boundary in order to allow for the discontinuities of the fluxes there [8]. The discretization of boundary and domain and the location of source points are described in Fig. 2. Uniform space discretization is used here.

To evaluate the coefficients of discretized equations, some integral formulae are used, which are presented in the Appendix, for integrations of the fundamental solution over boundary and internal elements. These formulae require much less computation time for the above process than numerical quadrature, e.g. Gauss–Legendre quadrature, and provide the evaluations with high accuracy.

In order to improve the accuracy of solution, a technique locating the source points not on the boundary but inside and outside the domain is introduced. This technique is called the direct regular method (DRM), while the usual boundary element technique, which locates the source points on the boundary and in the domain is called the direct singular method (DSM) [13, 14]. The singular integrals encountered in the boundary integrations of fundamental solution in the case of DSM do not occur in the case of DRM. It is considered that this is one of the reasons that the accuracy of solution can be improved by DRM. In the location of source points for present analysis, the distances between the boundary and the source points outside the domain are once or $\sqrt{2}$ times as long as the length of a boundary element. Those distances are determined by following the numerical investigation about the variation of accuracy with the location for elastostatic problems by Yuuki *et al.* [13, 14]. To show the improvement of the accuracy by DRM for heat conduction problem, we take a 1D linear problem as an example;

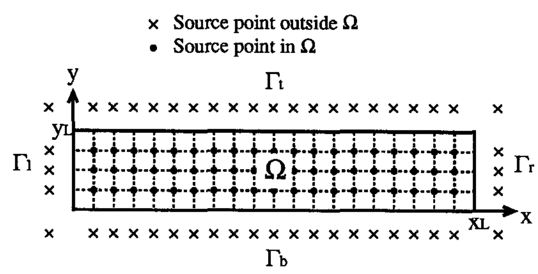

Fig. 2. Domain under consideration and its discretization.

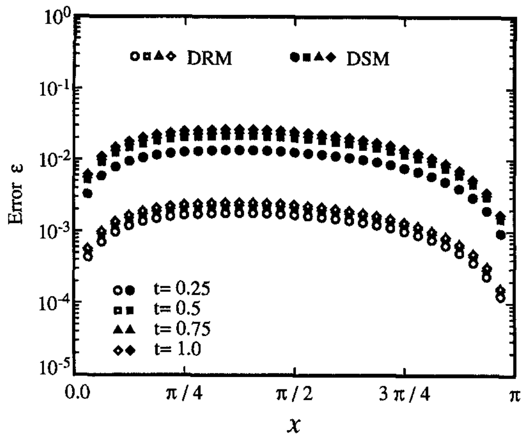

Fig. 3. Distributions of errors of DRM and DSM for a linear problem.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{14}$$

$$T = x+1+\sin x \quad \text{at} \quad t = 0 \tag{15}$$

$$T = 1 \quad \text{on} \quad x = 0 \tag{16}$$

$$T = \pi+1 \quad \text{on} \quad x = \pi. \tag{17}$$

This has an exact solution

$$T = x+1+e^{-t}\sin x. \tag{18}$$

This problem is solved as an equivalent 2D problem with the additional boundary condition,

$$\frac{\partial T}{\partial n} = 0 \quad \text{on} \quad \Gamma_b \cup \Gamma_t. \tag{19}$$

Equally spaced 33×5 nodes and time step, $\tau = 1/32$ are used for numerical solution for both DRM and DSM. The behaviour of accuracy improvement is shown in Fig. 3. The error ε is defined as

$$\varepsilon = \left|\frac{T^{\text{num}} - T^{\text{exact}}}{T^{\text{exact}}}\right|. \tag{20}$$

The error of DRM is about one tenth as much as that of DSM. The similar behaviour of the improvement of accuracy by DRM can be observed for nonlinear problems.

4. ILLUSTRATIVE EXAMPLES

In order to demonstrate the validity of the present BIEM, two examples are considered. There are few nonlinear problems that have the known exact solution. Among them is the first example (example 1). The result computed by the present method is compared with its exact solution. The second example (example 2) is a problem taken by Chen and Lin [15]. This is a nonlinear problem with the thermal conductivity and specific heat of linear variation with temperature. They solved it by FDM and FEM with hybrid Laplace transform technique and the Crank–Nicolson scheme, and reported the results. In each example, nonlinear equations discretized by the present BIEM are solved numerically by Newton's iteration method. One to three iterations are needed for convergence.

Example 1 is the problem given by

$$\frac{\partial T}{\partial t} = T\frac{\partial^2 T}{\partial x^2} \tag{21}$$

with initial and boundary conditions,

$$T(x,0) = 1-\tfrac{1}{2}x^2 \tag{22}$$

$$\frac{\partial T}{\partial n}(0,t) = 0 \tag{23}$$

$$\frac{\partial T}{\partial n}(1,t) = -\frac{1}{1+t}. \tag{24}$$

The equivalent equation,

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{25}$$

is considered in the place of equation (21). An exact solution of this problem is

$$T(x,t) = \frac{1}{1+t}(1-\tfrac{1}{2}x^2). \tag{26}$$

This problem is also considered as a 2D one in a same manner as the linear example in the previous section. Equally spaced 33×5 nodes and time step, $\tau = 1/32$ are used again.

The constant C_r introduced in equation (6) is chosen as 3.0 for this example. This value is determined by the following consideration. In the space-time domain under consideration, $[0,1] \times [0,1]$, we have $1/4 \leqq T \leqq 1$, and therefore, $1 \leqq C(T) = 1/T \leqq 4$. So the value of C_r is selected as 3.0, which falls within the range where the last inequalities hold. Since the maximum and the minimum of temperature can be estimated a priori from the initial and the boundary conditions for most heat conduction problems, this way of determination of C_r can be generally applied.

Figure 4 shows the comparison of the present com-

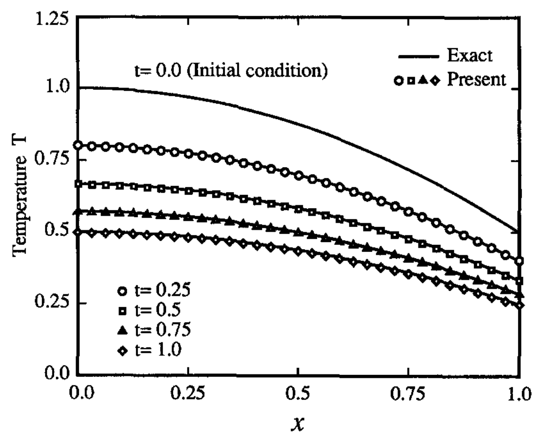

Fig. 4. Comparison of the present and the exact solutions of example 1.

putational result with the exact solution. It is seen that the present result is in good agreement with the exact solution. It implies the validity of the present BIEM.

Example 2 is given by the equation,

$$c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) \tag{27}$$

where

$$c(T) = 1+\beta T \tag{28}$$

and

$$k(T) = 1+\kappa T. \tag{29}$$

The coefficients β and κ are constants. The initial and the boundary conditions,

$$T = 0 \quad \text{at} \quad t = 0 \tag{30}$$

$$T = 1 \quad \text{on} \quad x = 0 \tag{31}$$

$$T = 0 \quad \text{on} \quad x = 1 \tag{32}$$

are imposed. By performing the Kirchhoff's transformation (2),

$$u(T) = \int_{T_0}^{T} k(s)\,\mathrm{d}s = T+\frac{\kappa}{2}T^2 \tag{33}$$

where $T_0 = 0$, equations (27) and (30)–(32) are transformed to

$$C(u)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{34}$$

where $C(u) = c(T)/k(T)$ and

$$u = 0 \quad \text{at} \quad t = 0 \tag{35}$$

$$u = 1+\frac{\kappa}{2} \quad \text{on} \quad x = 0 \tag{36}$$

$$u = 0 \quad \text{on} \quad x = 1 \tag{37}$$

respectively. Additional boundary condition (19) is transformed to

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_b \cup \Gamma_t. \tag{38}$$

The present numerical solutions, the solutions obtained by the conventional forward-time centered-space explicit finite difference scheme and the fully implicit centered-space finite difference scheme, and the results by the Crank–Nicolson scheme and the FEM with hybrid Laplace transform (HLT FEM) that are reported by Chen and Lin [15], are shown in Tables 1–3. The constant C_r for the present method is chosen as 0.75 for all cases. In each case, 21 × 5 uniformly located nodes are used for the present BIEM, uniform 21 nodes for the implicit FDM and the Crank–Nicolson scheme, uniform 11 nodes for the HLT FEM, and uniform 41 nodes for the explicit FDM. The time steps for each method are indicated in the tables. The HLT technique does not need time-step algorithm to reach a specific time [15]. It is convenient to obtain the solution at a time, but is not

Table 1. Comparison of the results of example 2 for $\kappa = 1.0$ and $\beta = 0.5$

x	Present BIEM	Explicit FDM	Crank–Nicolson	Implicit FDM	Implicit FDM	HLT FEM	Present BIEM	Explicit FDM	Crank–Nicolson	Implicit FDM	Implicit FDM	HLT FEM
τ	1/30	1/4110	1/100	1/300	1/100	—	1/30	1/4110	1/100	1/300	1/100	—
	$t = 0.1$						$t = 1.0$					
0.2	0.7390	0.7406	0.7337	0.7363	0.7283	0.7348	0.8439	0.8439	0.8439	0.8439	0.8439	0.8439
0.4	0.4863	0.4830	0.4705	0.4762	0.4641	0.4718	0.6733	0.6733	0.6733	0.6733	0.6733	0.6733
0.6	0.2752	0.2645	0.2513	0.2587	0.2495	0.2509	0.4832	0.4832	0.4832	0.4832	0.4832	0.4832
0.8	0.1191	0.1086	0.1003	0.1059	0.1021	0.0986	0.2649	0.2649	0.2649	0.2649	0.2649	0.2649

Table 2. Comparison of the results of example 2 for $\kappa = 0.5$ at $t = 0.5$

x	Present BIEM	Explicit FDM	Crank–Nicolson	Implicit FDM	HLT FEM	Present BIEM	Explicit FDM	Crank–Nicolson	Implicit FDM	HLT FEM
τ	1/32	1/8330	1/100	1/300	—	1/32	1/3120	1/100	1/300	—
			$\beta = -0.5$					$\beta = 1.0$		
0.2	0.8283	0.8283	0.8283	0.8283	0.8283	0.8247	0.8234	0.8230	0.8230	0.8219
0.4	0.6455	0.6455	0.6455	0.6455	0.6456	0.6394	0.6371	0.6365	0.6364	0.6347
0.6	0.4492	0.4492	0.4492	0.4492	0.4494	0.4427	0.4404	0.4397	0.4396	0.4377
0.8	0.2359	0.2359	0.2359	0.2358	0.2360	0.2315	0.2300	0.2296	0.2295	0.2283

Table 3. Comparison of the results of example 2 for $\beta = 0.5$ at $t = 0.5$

x	Present BIEM	Explicit FDM	Crank–Nicolson	Implicit FDM	HLT FEM	Present BIEM	Explicit FDM	Crank–Nicolson	Implicit FDM	HLT FEM
τ	1/32	1/3120	1/100	1/300	—	1/32	1/4090	1/100	1/300	—
			$\kappa = -0.5$					$\kappa = 1.0$		
0.2	0.7238	0.7201	0.7192	0.7195	0.7156	0.8431	0.8431	0.8430	0.8429	0.8431
0.4	0.5021	0.4976	0.4970	0.4968	0.4918	0.6719	0.6718	0.6718	0.6715	0.6718
0.6	0.3144	0.3108	0.3101	0.3102	0.3060	0.4817	0.4815	0.4815	0.4812	0.4815
0.8	0.1494	0.1476	0.1469	0.1473	0.1451	0.2637	0.2636	0.2636	0.2634	0.2636

suitable to trace the transient behaviour of the problem. The explicit FDM is employed to obtain the reference solutions that can be thought to be close enough to the exact solutions. The time steps of it are determined empirically so that the computation can be performed stably and accurately. For example, the determined time steps, 1/4110 and 1/8330, can not be replaced with 1/4100 and 1/8320, respectively without loss of stability and accuracy. The time step of the implicit FDM, $\tau = 1/100$ in Table 1, is determined by taking account of that of the Crank–Nicolson scheme taken by Chen and Lin. The result of the implicit FDM with $\tau = 1/100$ at $t = 0.1$ deviates considerably from the other results that are in good agreement with one another, while its computation is stable. The stability condition of the implicit scheme is usually much weaker than that of the explicit scheme, but the time step can not be taken so large with holding the accuracy of the computation. In order to obtain the results as accurate as the other methods, $\tau = 1/300$, is taken for the implicit FDM as shown in Table 1, and is used as well for the cases shown in Tables 2 and 3.

In all tables, no significant difference is seen between the results by the present BIEM and the results by the explicit FDM and other methods. It is suggested that still larger time step can be taken for the present method for the nonlinear problems than for the other methods. This advantage has been known for BEM for linear heat conduction problem [5, 6] and it seems to hold for the nonlinear problems.

5. CONCLUSION

A boundary integral equation formulation for transient nonlinear heat conduction problems with temperature-dependent material properties is proposed. It is a direct extension of BEM for linear heat conduction problems and is easy to numerically implement and apply to 2D and 3D problems. Since it does not require any modelling, the results can retain the nonlinear characteristics of the problem, which may be lost by linear modellings.

DRM is employed for the improvement of accuracy of solution. It is shown that, for the analysis of heat conduction problems, the improvement by DRM is significant.

It can be seen, from the examples, that the present formulation with boundary element techniques and DRM is valid enough to trace the transient nonlinear thermal response. Larger time steps than those of the other methods can be taken to proceed the computation stably and accurately as usual BEM can for linear problems.

Some integral formulae are used for the calculation of the coefficients of the discretized equations of 2D problem. Improvements of both the efficiency and the accuracy of the computation are achieved by using these formulae. By numerical quadrature, it takes a long time to calculate the coefficients of the equations derived by BEM and BIEM formulation, and the re-calculation of them, which is required, for example, in the case of the change of time step τ, increases the total computation time of the analysis beyond practicality of numerical solution. Therefore, it has been considered to be unrealistic with any numerical quadratures so far. The great deal of reduction of computation time by using the integral formulae leads us to the possibility of the practical re-calculation. In fact, the computation time for the calculation of the coefficients is as long as the time that takes to obtain the solution at one time step in the present analysis. Its increase of computation time is tolerable one.

As for the domain integrations, we present, in the Appendix, only formulae over rectangular domain. Although the integrations over triangular domains are required for many applications, analytical formulae for those integrations can not be obtained or are too lengthy to take the place of numerical quadratures. Therefore, some numerical quadrature has to be used for evaluations of the integrations over triangular domains for heat conduction problems.

REFERENCES

1. S. Orivuor, Efficient method for solution of nonlinear heat conduction problems, *Int. J. Numer. Meth. Engng* **14,** 1461–1476 (1979).
2. T. J. R. Hughes, Unconditionally stable algorithms for nonlinear heat conduction, *Comp. Meth. Appl. Mech. Engng* **10,** 135–139 (1977).
3. J. Kujawski, Analysis of the collocation time finite element method for the non-linear heat transfer equation, *Int. J. Numer. Meth. Engng* **24,** 103–107 (1987).
4. S. V. Patankar, *Numerical Heat Transfer and Fluid Flow*. Hemisphere, New York (1980).
5. C. A. Brebbia, J. C. F. Telles and L. C. Wrobel, *Boundary Element Techniques*. Springer, Berlin (1984).
6. L. C. Wrobel and C. A. Brebbia, A formulation of the boundary element method for axisymmetric transient heat conduction, *Int. J. Heat Mass Transfer* **24,** 843–850 (1981).
7. R. Bialecki, Solving nonlinear heat transfer problems by BEM. In *Boundary Elements X*, Vol. 2 (Edited by C. A. Brebbia), pp. 195–222. CMP Publications, Southampton (1988).
8. N. Zabaras and S. Mukherjee, An analysis of solidification problems by the boundary element method, *Int. J. Numer. Meth. Engng* **24,** 1879–1900 (1987).
9. C. P. Hong, T. Umeda and Y. Kimura, Numerical models for casting solidification—I. The coupling of the boundary element and finite difference methods for solidification problems, *Metall. Trans. B* **15B,** 91–99 (1984).
10. C. P. Hong, T. Umeda and Y. Kimura, Numerical models for casting solidification—II. Application of the boundary element method to solidification problems, *Metall. Trans. B* **15B,** 101–107 (1984).
11. P. Skerget and C. A. Brebbia, Time dependent nonlinear potential problems. In *Topics in Boundary Element Research*, Vol. 2 (Edited by C. A. Brebbia), pp. 63–86. Springer, Berlin (1985).
12. M. Kikuta, H. Togoh and M. Tanaka, A boundary element method for nonlinear transient heat conduction problems. In *Boundary Elements VIII*, Vol. I (Edited by M. Tanaka and C. A. Brebbia), pp. 47–58. Springer, Berlin (1986).
13. R. Yuuki, T. Matsumoto and H. Kisu, The usefulness and limit of the direct regular method in boundary element elastostatic analysis, *JSME Int. J.* **30,** 227–233 (1987).
14. R. Yuuki, G. Q. Cao and K. Maekawa, Accurate BEM elastostatic analysis for very slender body and thin plate, *Proceedings of IABEM '91* Kyoto (Edited by S. Kobayashi and N. Nishimura). Springer, Berlin (1991).
15. H.-T. Chen and J.-Y. Lin, Hybrid Laplace transform technique for non-linear transient thermal problems, *Int. J. Heat Mass Transfer* **34,** 1301–1308 (1991).
16. T. Goto and M. Suzuki, A boundary element approach for nonlinear heat conduction problems, *Proceedings of 4th BEM Technology Conference* (Edited by Japan Society for Computational Methods in Engineering), pp. 1–6 (1994).
17. T. Goto and M. Suzuki, Boundary element solution of a latent heat model of heat conduction problem with phase change, *Proceedings of 3rd BEM Technology Conference* (Edited by Japan Society for Computational Methods in Engineering), pp. 1–6 (1993).

APPENDIX: INTEGRAL FORMULAE

We present the integral formulae that are used for integrations of the fundamental solution u^* of 2D heat conduction equation, defined by equation (7) in Section 2. Prior to it, we define an exponential integral function $E_1(x)$ and an error function $\mathrm{erf}(x)$ as

$$E_1(x) = \int_x^\infty \frac{\exp(-s)}{s}\,\mathrm{d}s = -\gamma - \ln|x| - \sum_{n=1}^{\infty}\frac{(-x)^n}{nn!}$$

and

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-s^2)\,\mathrm{d}s$$

where γ is Euler's constant. The evaluation of $\mathrm{erf}(x)$ can be performed easily and efficiently by making use of mathematical library prepared on most computers. Hereafter, a, b, c, d and C are real constants such that $ab \geqq 0$, $cd \geqq 0$ and $C > 0$.

Formulae used for boundary integrations of u^* are as follows.

$$\int_a^b E_1(C(x^2+y^2))\,\mathrm{d}x = bE_1(C(b^2+y^2)) - aE_1(C(a^2+y^2)) + \frac{\sqrt{\pi}}{\sqrt{C}}\exp(-Cy^2)[\mathrm{erf}(b\sqrt{C}) - \mathrm{erf}(a\sqrt{C})] - 2y^2\int_a^b \frac{\exp(-C(x^2+y^2))}{x^2+y^2}\,\mathrm{d}x \quad \text{(A1)}$$

$$\int_a^b E_1(Cx^2)\,\mathrm{d}x = bE_1(Cb^2) - aE_1(Ca^2) + \frac{\sqrt{\pi}}{\sqrt{C}}[\mathrm{erf}(b\sqrt{C}) - \mathrm{erf}(a\sqrt{C})] \quad \text{(A2)}$$

$$\lim_{x\to 0} xE_1(Cx^2) = 0 \quad \text{(A3)}$$

$$\int_a^b xE_1(C(x^2+y^2))\,\mathrm{d}x = \frac{1}{2C}[\exp(-C(a^2+y^2)) - \exp(-C(b^2+y^2))] + \tfrac{1}{2}[(b^2+y^2)E_1(C(b^2+y^2)) - (a^2+y^2)E_1(C(a^2+y^2))] \quad \text{(A4)}$$

$$\int_a^b xE_1(Cx^2)\,\mathrm{d}x = \tfrac{1}{2}b^2E_1(Cb^2) - \tfrac{1}{2}a^2E_1(Ca^2) + \frac{1}{2C}[\exp(-Ca^2) - \exp(-Cb^2)] \quad \text{(A5)}$$

$$\lim_{x\to 0} x^2E_1(Cx^2) = 0. \quad \text{(A6)}$$

Note that the numerical integration of integral term on the right-hand side of equation (A1) is easier than that of the left-hand side, and, due to the factor $2y^2$, the integral on the right-hand side does not encounter the difficulty of singularity. The logarithmic singularities of the left-hand sides of equations (A2) and (A5) are transferred into first and second terms on the right-hand sides of each equation and those singularities are resolved by equations (A3) and (A6).

Formulae used for integrations of u^* over rectangular domain are as follows.

$$\int_c^d \int_a^b \exp\left(-\frac{x^2+y^2}{C}\right) \mathrm{d}x\,\mathrm{d}y = \frac{\pi C}{4}\left[\operatorname{erf}\left(\frac{b}{\sqrt{C}}\right) - \operatorname{erf}\left(\frac{a}{\sqrt{C}}\right)\right]\left[\operatorname{erf}\left(\frac{d}{\sqrt{C}}\right) - \operatorname{erf}\left(\frac{c}{\sqrt{C}}\right)\right] \quad \text{(A7)}$$

$$\int_c^d \int_a^b x \exp\left(-\frac{x^2+y^2}{C}\right) \mathrm{d}x\,\mathrm{d}y = \frac{C\sqrt{(\pi C)}}{4}\left[\exp\left(-\frac{b^2}{C}\right) - \exp\left(-\frac{a^2}{C}\right)\right] \times \left[\operatorname{erf}\left(\frac{c}{\sqrt{C}}\right) - \operatorname{erf}\left(\frac{d}{\sqrt{C}}\right)\right] \quad \text{(A8)}$$

$$\int_c^d \int_a^b y \exp\left(-\frac{x^2+y^2}{C}\right) \mathrm{d}x\,\mathrm{d}y = \frac{C\sqrt{(\pi C)}}{4}\left[\exp\left(-\frac{d^2}{C}\right) - \exp\left(-\frac{c^2}{C}\right)\right] \times \left[\operatorname{erf}\left(\frac{a}{\sqrt{C}}\right) - \operatorname{erf}\left(\frac{b}{\sqrt{C}}\right)\right] \quad \text{(A9)}$$

$$\int_c^d \int_a^b xy \exp\left(-\frac{x^2+y^2}{C}\right) \mathrm{d}x\,\mathrm{d}y = \frac{C^2}{4}\left[\exp\left(-\frac{b^2}{C}\right) - \exp\left(-\frac{a^2}{C}\right)\right] \times \left[\exp\left(-\frac{d^2}{C}\right) - \exp\left(-\frac{c^2}{C}\right)\right]. \quad \text{(A10)}$$